Irreversible phase transitions in contact processes with Lévy exchanges and long-range interactions

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The contact process (CP) is generalized allowing the exchange of particles via Lévy flights, where the flying length (l) is a random variable with a probability distribution given by $P(l)\propto l^{-d-\sigma}$, where d is the spacial dimension and σ is the dimension of the random walk. The contact process with Lévy flights (CPLF) exhibits irreversible phase transitions between an active state and a vacuum state. It is show that within the superdiffusive regime of the walkers (i.e., $\sigma<1$), the Lévy mechanism effectively build up additional long-range correlations, therefore the critical exponents of the CPLF model depart from those of the standard CP and they are tunable functions of σ. Comparison of the critical exponents characteristic of branching annihilating Lévy walkers [E. Albano Europhys. Lett. **34**, 97 (1996)] and those of the CPLF gives strong evidences on a universality class which comprises second order irreversible phase transitions in systems involving Lévy exchanges and/or flights. It is suggested that the CPLF is equivalent to the standard CP with long-range interactions generated by a potential decaying with distance r as a power law of the form $V(r)\propto r^{-d-\sigma}$. [S1063-651X(96)03510-6]

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I. INTRODUCTION

Interest in the understanding of the behavior of far-from-equilibrium many-particle systems has recently experienced a rapid growth because it is relevant in many branches of science such as physics, chemistry, biology, ecology, and even sociology. Special attention has been devoted to irreversible systems exhibiting irreversible phase transitions (IPTs) from active (stationary) to inactive states. A common feature of such systems is that they evolve according to a Markov process governed by local, intrinsically irreversible transitions rules; such models are collectivelly known as interacting particle systems [1,2]. Some examples are the contact process (CP) [1,3–7], the A model [3], surface reaction models (see, e.g., [8–13], etc.), directed percolation [14], forest-fire models with immune trees [15], the stochastic game of life [16], branching annihilating random walkers [17–19], etc. So far in all these examples long-range correlations are developed as a consequence of the microscopic mechanisms governing the evolution of the systems. In fact, in most cases the ''potential energy'' of interaction between particles (or individuals) is simply ignored, while in other examples only short-range interactions are considered [13,20]. Therefore, our understanding of IPTs in systems with long-range interactions is restricted to the same scarce analytical results [2]. The lack of computer simulations in this field is probably due to the huge effort required to obtain accurate critical exponents.

Recently, it has been demonstrated, in the field of reversible phase transitions, that random exchange via Lévy flights can effectively generate long-range interactions [21,22]. The Lévy flight [23,24] is a random walk in which the step length (l) is a random variable with a probability distribution given by

$$P(l)\propto l^{-d-\sigma}, \tag{1}$$

where d is the spacial dimension and the parameter σ is the dimension of the random walk for $0<\sigma<1$. It should be noted that within that range of σ the walker exhibits superdiffusive behavior, while for $\sigma=1$ one recovers ordinary diffusion [25]. So, in Ising-like models, the random Lévy exchange of spins generates an effective interaction potential decaying with distance r as a power law of the form [21,22]

$$V(r)\propto r^{-d-\sigma}. \tag{2}$$

Within this context, the aim of the present work is to study, by means of computer simulations, the critical behavior of both a CP with Lévy exchanges (i.e., the CPLF model) and a CP with long-range interactions between particles (i.e., the CPLRI model). This study will contribute to the understanding of both, irreversible reaction processes with anomalous diffusion and IPTs in the presence of long-range exchanges and interactions. The manuscript is organized as follows: Sec. II gives brief details of the simulation, in Sec. III the theoretical background of the epidemic analysis used to study the dynamic critical behavior of the models is discussed and the obtained results are presented. The order parameter critical exponent is evaluated in Sec. IV. The obtained results are compared with recent data corresponding to branching annihilating walkers where the walkers (or the offspring) have a finite probability to undergo Lévy jumps in Sec. V; and finally our conclusions are stated in Sec. VI.

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II. THE MODEL AND DETAILS ON THE SIMULATION METHOD

Lévy flights

According to Eq. (1) the Lévy flight has a finite (although small) probability to perform rather long jumps ($l \to \infty$), however in nature actual random walks necessarily perform bounded hoppings (for experimental realizations of Lévy walkers see [26,27]). For this reason and also due to obvious limitations in the computer implementation of the algoritms, we have earlier introduced the bounded Lévy flights [28] (also named truncated Lévy flights [29]). So, the probability distribution of the hoppings is now given by

$$P(l) \propto l^{-d-\sigma}, \quad 0 \leqslant \sigma; \quad l \leqslant R_M, \tag{3}$$

where R_M is the length of the longest possible flight. In the simulations we have used $R_M = 10^4$ which is a quite safe approach since in the worse case ($\sigma = 0$) the probability of having jumps longer than 10^3 lattice spaces is negligible.

B. The standard contact process (CP)

The contact process (CP), as proposed by Harris [30], is a model for the growth of an epidemic with a single species. The system evolves via a Markov process consisting in a sequence of elementary transitions, each involving a single process which takes place at a randomly selected site. In the CP each site i of a lattice, in d dimensions, is either vacant or occupied (denoted by $\gamma_i = 0$, or $\gamma_i = 1$, respectively). Multiple occupancy of lattice sites is forbidden. Randomly selected particles are annihilated spontaneously with probability p independent of the state of the others. However, a randomly chosen vacant site becomes occupied with probability n/z, where n is the number of occupied nearest neighbors and z is the coordination number of the lattice. Since spontaneous creation of particles is not allowed, the system can irreversibly evolve into a vacuum state, which is the absorbing state of the Markov process. If p is very large the system always enters the absorbing state, but for small enough values of p the system has an active state with nonzero average particle density ρ. In one dimension the system undergoes an irreversible phase transition (IPT) from the active state to the absorbing state at a critical probability p_c. The IPT is continuous (second order) and belongs to the universlity class of directed percolation.

C. The CP with Lévy exchanges (CPLF)

The generalization from the CP to the CPLF is straightforward: a randomly selected empty site evolves according to the rules of the CP, however, a randomly chosen occupied site may either evole according to the rules of the CP with probability $1-\tau$ or undergo a Lévy exchange with other site with probability τ. We carried out Monte Carlo simulations of the CPLF process in one dimension for $\tau = 1/2$ and $0 \leqslant \sigma \leqslant 11$.

D. The CP with long-range interactions (CPLRI)

In the CPLRI a randomly selected occupied site is vacated with probability p; as in the CP; while a randomly empty sites becomes occupied with probability

$$P(\gamma_i = 0 \to \gamma_i = 1) = C \sum_{j \neq i} \gamma_j |i-j|^{-1-\sigma}, \tag{4}$$

where C is the normalization constant so that a single vacant site in an otherwise filled lattice becomes occupied with unitary probabilty.

III. EPIDEMIC ANALYSIS

A. Theoretical background and simulation details

Test runs of both the CPLF and the CPLRI models show that, in fact, each system reaches an active stationary state for small enough values of p, while increasing p causes the system to irreversibly evolve into an absorbing (vacuum) state. The IPTs are continuous (second order) and the critical values of p at which such transitions take place depend on σ. Determining critical behavior from steady state simulations in irreversible dynamic systems is often very difficult due to large fluctuations, finite size effects, critical slowing down, and uncertainty in the location of the critical point. In fact, working with finite lattices, due to fluctuations of the stochastic process, there is always a finite probability of the active state to become inactive. Furthermore, this probability increases when approaching the critical edge. These shortcomings can be avoided performing time dependent simulations also known as epidemic analysis. This kind of simulations allow us to determine reliable critical exponents related to the dynamical critical behavior of the system under consideration. The general idea behind epidemic simulations is to start from a configuration which is very close to the inactive state, and follow the averaged time evolution of this configuration by generating a large ensemble of independent realizations. So, the epidemic analysis is performed as follows: one starts, at $t = 0$, with two occupied nearest neighbor sites, placed in the center of the lattice, in an otherwise empty sample. Then the configuration is allowed to evolve according to the rules of the model. As the number of active sites is always rather small, an efficient algorithm can be devised by keeping two lists: one containing the occupied sites and the other with the empty sites which have at least one occupied nearest neighbor site. In each elementary step a site of those lists is chosen randomly. After each attempted change the time is incremented by $1/[N(t) + N_e(t)]$, where $N(t)$ and $N_e(t)$ are the number of occupied and empty sites contained in the lists at time t. Thus one Monte Carlo time step (MCts) equals, on the average, one attempted update per active site.

The time evolution of the sites is monitored and the following quantities are computed: (i) The average number of occupied sites $N(t)$, (ii) the survival probability $P(t)$ (i.e., the probability that the system had not entered in the inactive state at time t), and the average mean square distance of spreading from the center of the lattice $R^2(t)$ [distances are meassured in lattice units (LU)]. Notice that $N(t)$ is averaged over all runs whereas $R^2(t)$ is only averaged over the surviving runs.

At criticality, the following scaling behavior is expected to hold [14]

$$P(t) \propto t^{-\delta}, \tag{5}$$

TABLE I. Critical points and critical exponents of CPLF, CPLRI, branching annhilating Lévy flights (BALF, taken from Ref. [31]) and directed percolation (DP, taken from Ref. [14]). The last column is a test of the validity of the scaling relationship given by Eq. (11). Figures between parenthesis indicate the error bars in the last digit.

Model	σ	p_c	η	δ	z	$dz-2\eta-4\delta$
DP	—	—	0.308	0.160	1.265	0.009
CPLF	11	0.4235(5)	0.305(5)	0.161(3)	1.257(5)	0.003
CPLF	2	0.4380(5)	0.304(5)	0.166(3)	1.261(5)	−0.011
CPLF	1.50	0.4490(5)	0.306(5)	0.166(3)	1.260(5)	−0.016
CPLF	1	0.4710(5)	0.306(5)	0.165(3)	1.260(5)	−0.012
CPLF	0.75	0.4890(5)	0.328(5)	0.159(3)	1.262(5)	−0.030
CPLF	0.50	0.5137(3)	0.352(5)	0.145(3)	1.265(5)	−0.019
CPLF	0.25	0.5463(3)	0.367(5)	0.14(1)	1.28(2)	−0.014
CPLF	0.0	0.5868(3)	0.403(8)	0.12(1)	1.30(2)	0.014
CPLRI	0.0	0.400(5)	0.31(1)	0.16(1)	1.26(2)	0.00
CPLRI	1.0	0.593(3)	0.31(1)	0.16(1)	1.27(2)	0.01
BALF	11	0.1070(5)	0.308(5)	0.156(3)	1.251(5)	0.011
BALF	2	0.1205(5)	0.303(5)	0.158(3)	1.258(5)	0.02
BALF	1.50	0.1306(3)	0.305(5)	0.164(3)	1.262(5)	−0.004
BALF	1	0.1563(3)	0.309(5)	0.163(3)	1.263(5)	−0.007
BALF	0.75	0.1861(3)	0.324(5)	0.164(3)	1.265(5)	−0.039
BALF	0.50	0.2475(5)	0.351(5)	0.150(3)	1.270(5)	−0.032
BALF	0.25	0.3853(3)	0.366(5)	0.13(1)	1.28(2)	0.005
BALF	0.0	0.6598(3)	0.405(8)	—	—	—

$$N(t)\propto t^{\eta}, \tag{6}$$

and

$$R^2(t)\propto t^{z}, \tag{7}$$

where δ, η, and z are dynamic critical exponents. At criticality, one expects that log-log plots of $P(t)$, $N(t)$, and $R^2(t)$ versus t would give straight lines, while upward and downward deviations will occur even slightly off criticality. This behavior would allow a precise determination of both the critical points and the critical exponents.

After determining the critical points, one can gain further insight of the critical behavior of the model performing epidemic analysis within the subcritical (vacuum) state. In fact close to the critical point the following scaling law should hold [14]:

$$N(t)\propto t^{\eta}\Psi(|p-p_c|t^{1/\nu_{\parallel}}), \tag{8}$$

where $\nu_{\parallel}$ is the correlation length exponent in the so called time direction. In the vacuum state the correlations are short-ranged and one therefore expects $N(t)$ to decay exponentially. This can only happen if for $\Delta p=p-p_c\rightarrow 0$ and $t\rightarrow\infty$, the scaling function Ψ behaves as

$$\Psi(y)\propto(y)^{-\eta\nu_{\parallel}}\exp^{-(y)^{-\nu_{\parallel}}}. \tag{9}$$

Therefore, using Eqs. (7) and (8) it follows:

$$N(t)\propto(\Delta p)^{-\eta\nu_{\parallel}}\exp^{-(\Delta p)^{\nu_{\parallel}}t}. \tag{10}$$

It should be noted that the dynamic exponents are not fully independent but a number of scaling relations are expected to hold [14]. For example, the relationship

$$dz=2\eta+4\delta, \tag{11}$$

may be valid in d dimensions.

Results and discussion

The CPLF model. For very large values of σ, Lévy flights are restricted to nearest neighbor jumps since the probability of larger jumps is negligible. In this limit, the CPLF model is expected to exhibit the same critical behavior than the CP. In fact, test runs performed for $\sigma=11$ give critical exponents which are in excellent agreement with the universality class of directed percolation (see Table I). Decreasing σ causes the critical point to increase, but the exponents remain almost unchanged for $\sigma\geqslant 1$. This behavior can be understood since within that range of σ values, Lévy flights exhibit ordinary diffusion properties. However, a further decrease of σ causes the exponents to change (see Table I), in agreement with the fact that for $\sigma<1$ one has superdiffusive behavior.

Figures 1(a)–1(c) show log-log plots of $N(t)$, $P(t)$, and $R^2(t)$ versus t obtained close to criticality for $\sigma=0.75$. The plots of $N(t)$ and $P(t)$ versus t are quite sensitive with respect to small changes of p, so they are used to determine the critical points and exponents. Error bars corresponding to the critical points are estimated considering the closest values of p, such as off-critical behavior is observed. The exponents listed in Table I are obtained by means of least square fits of the asymptotic regime of plots like those shown in Fig. 1.

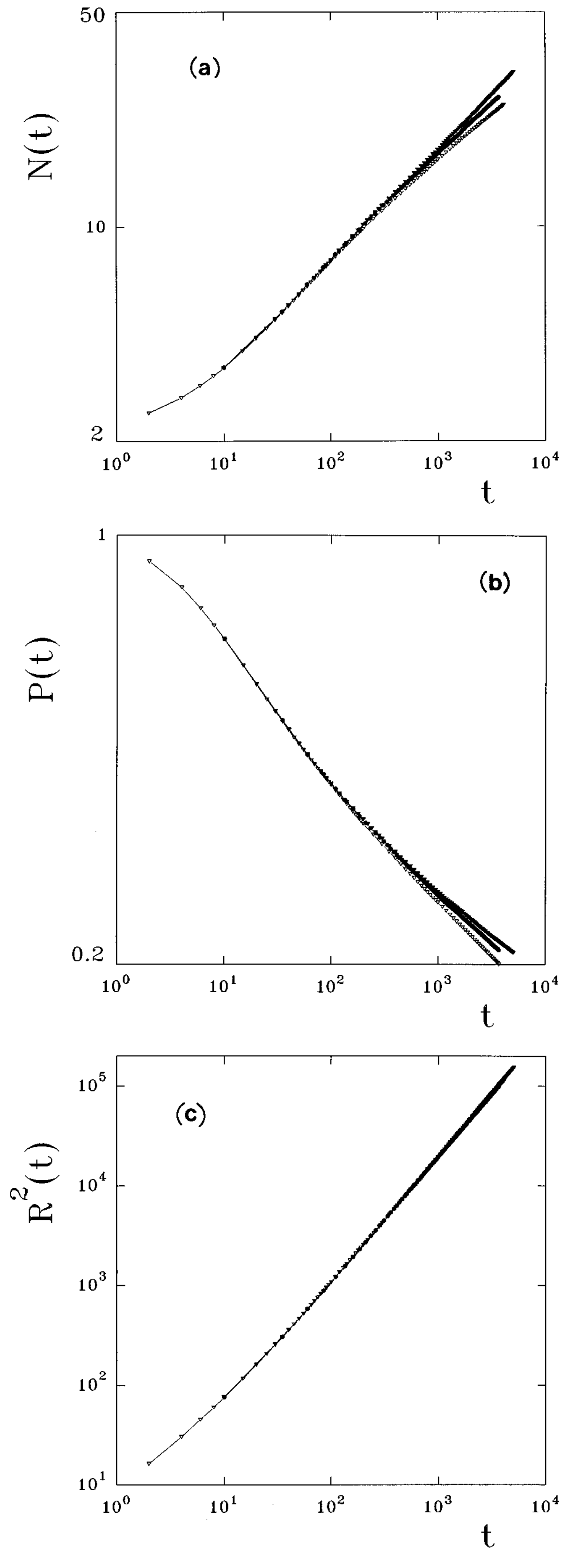

FIG. 1. Log-log plots of (a) the number of occupied sites $N(t)$; (b) the survival probability $P(t)$; and (c) the average square distance of spreading (measured in LU^2) $R^2(t)$ vs time t (measured in MCts), obtained close to criticality for $\sigma=0.75$. Upper curves: $p=0.4885$ (supercritical), medium curves: $p=0.4890$ (critical) and lower curves $p=0.4895$ (subcritical).

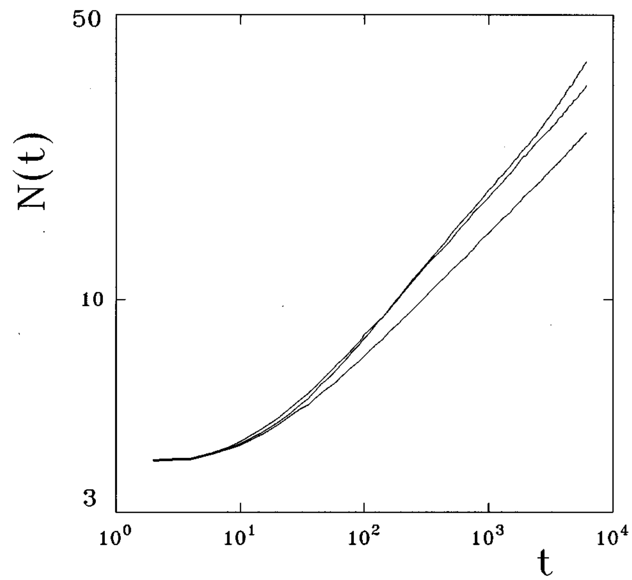

FIG. 2. Log-log plots of the number of occupied sites $N(t)$ vs time t (measured in MCts), obtained at criticality for different values of σ. Upper curve $\sigma=0.0$, medium curve $\sigma=0.50$, and lower curve $\sigma=11.0$.

Errors bars in the exponents are estimated by evaluating the slopes of the curves between different time intervals within the asymptotic regime. It should be noticed that for $\sigma \leqslant 0.25$ the plots of both $P(t)$ and $R^2(t)$ exhibit pronunced curvature, so the obtention of accurate exponents becomes difficult. Figure 2 shows log-log plots of N(t) versus t taken for different values of σ. Here the change in the asymptotic slope can clearly be observed.

The evaluated exponents allow us to test the scaling relationship given by Eq. (10) and derived for a standard directed percolation process [14]. The data show that the relationship holds, within error bars, for all the range of σ values covered by the study. As it also follows from Table I, the validity of Eq. (10) may be due to the operation of an interesting compensation effect: while z remains almost unchanged, η increases and δ decreases. At a given asymptotic time, a larger η value means that the number of occupied sites is also larger, so one should expects an increment of the spreading distance and consequently larger z values. However, since $R^2(t)$ is only averaged over surviving runs, this effect is canceled by the enhanced survivability of the occupied sites. The theoretical understanding of this behavior and the underlaying physics remains unclear.

From Eq. (10) it follows that in the subcritical regime $N(t)$ should decay exponentially and that the decay constant λ, governing the long-time behavior is proportional to $(\Delta p)^{\nu_\parallel}$. The model has been simulated in the subcritical region for different values of σ. Figure 3(a) shows that in ln-*linear* plots of $N(t)$ versus t one can see asymptotically a straight line behavior with slope λ. In fact, this statement is confirmed in Fig. 3(b) where log-log plots of λ versus Δp give straight lines and from the respective slopes one can evaluate the exponent $\nu_\parallel$. For $\sigma=2$ and $\sigma=1$ we have obtained $\nu_\parallel \cong 1.742(9)$ and $\nu_\parallel \cong 1.722(9)$, respectively. Large error bars are mostly due to uncertainties in the location of the critical point. The obtained figures are in excellent agree-

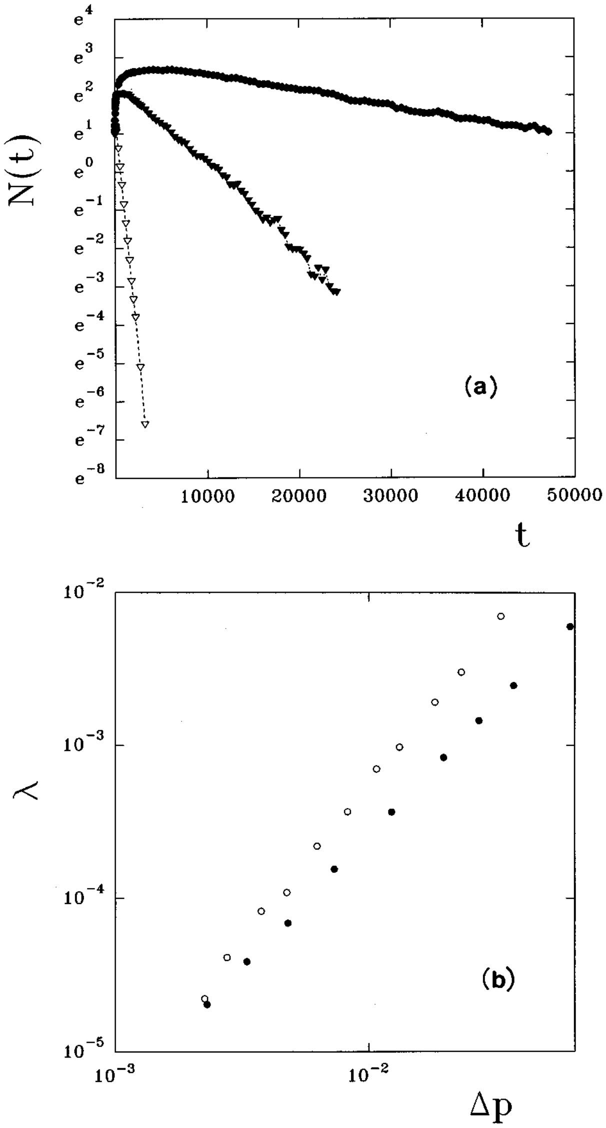

FIG. 3. (a) ln-lineal plots of the number of occupied sites $N(t)$ vs time t (measured in MCts), obtained within the subcritical regime for $\sigma=0.0$ and different values of p. Upper curve $p=0.5895$, medium curve $p=0.5930$, and lower curve $p=0.6100$. The critical probability is $p_c=0.586\,75$. (b) Log-log plots of λ vs Δp. Upper curve $\sigma=0.00$, the straight line has slope $\nu_{\parallel}=1.996$; lower curve $\sigma=2.00$, the straight line has slope $\nu_{\parallel}=1.722$.

ment with the accepted value for directed percolation in $1+1$ dimensions, i.e., $\nu_{\parallel}\cong 1.733$ [32]. However, decreasing σ departure from directed percolation values are found, e.g., for $\sigma=0$ we obtained $\nu_{\parallel}\cong 1.996(9)$. This finding is again in agreement with the superdiffusive behavior of the Lévy walkers observed for $\sigma<1$.

The CPLRI model. Extensive simulations of the epidemic behavior of the CPLRI model are only possible for $\sigma\geqslant 1$. The obtained results are listed in Table I. Since for this range of σ the interactions are almost restricted to nearest neighbor sites, it is not surprising that the obtained exponents are in agreement with those of directed percolation and the CPLF with $\sigma\geqslant 1$. For smaller values of σ the CPU time required largely exceds our capabilities. So, we are unable to test conclusively if both versions of the contact process are in the same universality class.

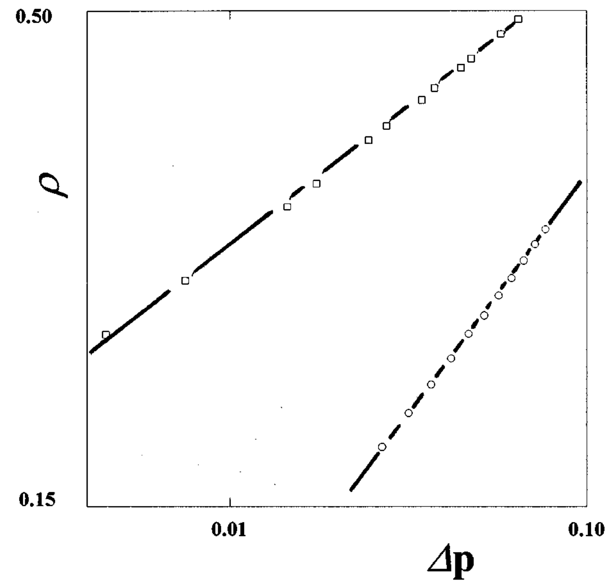

FIG. 4. Log-log plots of ρ (measured as number of occupied sites per LU) vs Δp, obtained for different values of σ. Lower curve $\sigma=0.0$, the straight line has slope $\beta=1/2$; and upper curve $\sigma=2.0$, the straight line has slope $\beta=0.278$.

IV. DETERMINATION OF THE ORDER PARAMETER CRITICAL EXPONENT

As it has already been discussed in the preceding section, the determination of critical exponents from steady state simulations in irreversible dynamic systems is often very difficult. However, due to the accuracity obtained in the evaluation of p_c an attempt has been made for the evaluation of the order parameter critical exponent. It is accepted that for irreversible dynamic process undergoing IPTs the natural order parameter is the concentration of active sites ρ, such as $\rho\rightarrow 0$ for $p\rightarrow p_c$. So, approaching criticality from the active state one has

$$\rho \propto (p_c - p)^{\beta}, \tag{12}$$

where β is the order parameter critical exponent. Figure 4 shows log-log plots of ρ versus $\Delta p = p_c - p$ obtained for the CPLF model using the values of p_c listed in Table I and taken two different values of σ. For $\sigma=2$ the slope of the straight line gives $\beta\cong 0.278(9)$ in good agreement with the best estimate for directed percolation in $(1+1)$ dimensions, i.e., $\beta=0.2763(6)$ (where the error bars account for β values determined using different lattices and for bond and site directed percolation) [33]. As expected, within the superdiffusive regime of the Lévy walkers the value of the exponents depart from standart directed percolation given, e.g., $\beta\cong 0.500(9)$ for $\sigma=0$. This result may suggest the rational value $\beta=1/2$, however, that conjecture can not strongly be supported because the large error bars which are due to the fact that the data have to be measured slightly out of criticality due to the presence of fluctuations which drive the system into the vacuum state.

V. BRANCHING ANNIHILATING LÉVY FLIGHTS (BALF)

In the standard branching annihilating random walker process (BAW), a single random walk branches at some specified rate and two random walkers annihilate when they meet [17–19]. In the generalization from BAW to BALF, randomly selected particles perform Lévy flights instead of jumps to nearest neighbor sites [31]. An epidemic analysis, like that performed in the present work, has allowed us to evaluated the relevant critical exponents which are listed in Table I for the sake of comparison. As can be observed the obtained exponents for the CPLF and the BALF models are in excellent agreement. Furthermore, for the correlation length exponent of BALF with $\sigma=0$ the value $\nu_{\parallel}\cong 1.98(1)$ has been obtained [31]. This figure is also in excellent agreement with the result $\nu_{\parallel}\cong 1.996(9)$ obtained in the present work for the CPLF process ($\sigma=0$) and may suggest that the limiting value of such exponent would be $\nu_{\parallel}=2$, however, the confirmation of this conjecture deserves further studies.

VI. CONCLUSIONS

A contact process where particles have a finite probability to undergo Lévy exchanges (CPLF) is formulated and studied by means of numerical simulations in one dimension. The CPLF model exhibits irreversible phase transitions between an active stationary state and a vacuum (absorbing) state. Within the range of Lévy exponents ($\sigma>1$) which corresponds to standard diffusion the obtained critical exponents reveal that the CPLF model belongs to the universality class of directed percolation. This finding is in agreement with well established concepts of universal behavior: since exchanges are restricted to finite distances, the diverging correlation length remains as the only relevant length scale. However, for smaller σ values, when superdiffusive behavior is observed, the exchanges are no longer restricted to finite distances and additional long-range correlations can effectively be established. So, in these cases, one observes departure from the standard directed percolation behavior and the critical exponents depend on σ; in other words they can be tuned varying σ.

Numerical results suggest that in the limit $\sigma=0$ at least two critical exponents may adopt rational values, i.e., $\beta=1/2$ and $\nu_{\parallel}=2$.

The comparison of the obtained critical exponents for two models, the CPLF process and the BALF reaction, strongly suggests the existence of a more general universality class of directed percolation, i.e., all second order irreversible phase transitions in processes involving Lévy exchanges and/or flights may have the same critical exponents depending only on σ and the dimensionality. Our simulations do not allow us to confirm if the Lévy exchange mechanism can effectively simulate a long-range interactive potential, as in the case of reversible phase transitions. However, in our opinion, the confirmation of this open question will certainly stimulate further work due to its relevance in the study of far from equilibrium irreversible processes.

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